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Black hole thermodynamics

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Abstract : The idea that a black hole may have an entropy proportional to the horizon area was introduced in the seventies. An area dependence was also found in the entropy of matter in the background of a black hole. Recent work has been more in the domain of *extremal* black holes, where the behaviour is not quite the same.

Keywords : Entropy, area, extremal black hole

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1. Introduction

A black hole is classically thought of as a region of intense gravitational field from which no form of energy – not even light – can leak out. The best known example is the Schwarzschild black hole solution of Einstein's equation

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

in empty space ($T_{\mu\nu} = 0$). It is described by the metric

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2, \tag{2}$$

where the parameter M refers to the mass of a point source of gravitation located at the centre of coordinates. The spacetime has a *horizon* at $r = 2M$, which is a singularity of this coordinate system, but the curvature is not singular there, and regular coordinates (discovered by Kruskal) may be chosen. There is, however, a curvature singularity at $r = 0$, the location of the point source.

Another example is the Reissner – Nordström solution of the Einstein – Maxwell equations. The metric is given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1} dr^2 + r^2 d\Omega^2 \tag{3}$$

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and the electric field by

$$F_{tr} = \tag{4}$$

with M and Q denoting the mass and the charge respectively of a point source at the origin. There are horizons at

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \tag{5}$$

the one at r_+ being the one relevant for external observers. It is assumed that $M \geq Q$. If this inequality fails to hold, the singularity at the origin is naked, *i.e.*, it is not covered by a horizon, and one no longer has a black hole spacetime. The limiting case when $Q = M$ and $r_+ = r_-$ is referred to as the extremal case. There is again a curvature singularity at $r = 0$, which is to be regarded as the location of the point source.

Classically, a black hole is stable and though it may absorb anything, it does not radiate. So a black hole may, at that level, be assigned zero temperature and correspondingly zero entropy. But the situation changes when quantum field theory is brought in to describe the interaction of matter with a black hole background. A careful definition of the vacuum using regular (Kruskal) coordinates shows that there is a net flow of particles away from the black hole, with a *thermal spectrum* corresponding to a temperature determined by the black hole parameters like mass and charge.

The strong gravitational field may be imagined to produce a particle-antiparticle pair by polarizing the vacuum. If this occurs outside the horizon, one of the pair may fall in, with the other moving away to infinity, thus contributing to an outward flow.

If the process continues, it actually becomes faster. This is because a black hole gets *hotter* when it loses mass. The black hole is thus expected to evaporate completely ! A regular spacetime should be left behind, with all the matter moving away.

This process of evaporation leads to a puzzle. A black hole can be imagined to start off in a pure state. When it evaporates, there is only thermal radiation, which is in a mixed state. Such a transition would appear to be non-unitary and to involve loss of information. Can it really occur ?

It is to be hoped that some day a clear answer will emerge. One expects a full quantum theory of the black hole, when it is available, to help us understand the issue. Meanwhile, we shall review the results of semiclassical approaches to black hole thermodynamics. That black holes possess entropy has been known for nearly thirty years now. And entropy is, of course, a measure of *lack of information.*

2. Black hole temperature and entropy

2.1 Early ideas :

A precursor of the idea of entropy in the context of black holes was the so-called area theorem [1]. According to this theorem, the area of the horizon of a system of black holes always increases. As a simple illustration, we may recall that a Schwarzschild black hole of mass M has a horizon area proportional to M^2. If two black holes of masses M_1, M_2 attract each other and combine to form a black hole of mass $M_1 + M_2$, the area increases, because $(M_1 + M_2)^2 > M_1^2 + M_2^2$. This theorem is certainly reminiscent of the second law of thermodynamics and thus of entropy.

Some other observations made around that time were collected together into a set of *laws of black hole mechanics* analogous to the laws of thermodynamics [2]. In order to understand these, it is necessary to know what *surface gravity* means. For any static, spherically symmetric black hole, it is given by

$$\kappa = \frac{1}{\sqrt{g_{rr}}} \frac{d\sqrt{-g_{tt}}}{dr}\bigg|_{r=r_+}, \tag{6}$$

and for the special case of a Reissner – Nordström black hole,

$$\kappa = \sqrt{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)} \; \frac{d\sqrt{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)}}{dr}\Bigg|_{r=r_+}$$

$$= \frac{r_+ - r_-}{2r_+^2}. \tag{7}$$

Note that this expression vanishes in the extremal case where $r_+ = r_-$

A more general definition can be given to accommodate rotating black holes, but we shall not go into it. For a spherically symmetric black hole, the surface gravity is obviously a constant, but it is true more generally that the surface gravity k remains constant on the horizon of a black hole. This is the *zeroeth law of black hole mechanics.* The *first law* states that

$$\frac{\kappa dA}{8\pi} = dM - \phi dQ, \tag{8}$$

where A represents the area of the horizon and ϕ the potential at the horizon. For the Reissner – Nordström black hole, where

$$\phi = Q/r_+, \quad A = 4\pi r_+^2, \tag{9}$$

one may easily verify (8). In view of the area theorem already stated, which is incorporated into this scheme as *the second law,* it is apparent that the surface gravity and the horizon area play the rôles of temperature and entropy, respectively. When these observations were first made, there was no obvious connection with thermodynamics, it was only a matter of analogy. But it was soon realized [3] that the existence of a horizon imposes a limitation on the amount of information available and hence may lead to an entropy, which should then be measured by the geometric quantity associated with the horizon, namely its area. Thus, upto a factor, A should represent the entropy and $\frac{\kappa}{8\pi}$ the temperature.

At that stage, this interpretation of the laws of black hole mechanics could not be regarded as completely satisfactory, and in any case the undetermined factor left a question mark. Fortunately, the problem was solved very soon. It was discovered that ***quantum*** issues were involved.

2.2 Temperature and entropy : fixing the scale :

First we consider the question of a ***conical singularity*** on passing to imaginary time. The metric

$$ds^2 = dr^2 + r^2\, d\theta^2, \tag{10}$$

which describes the flat Euclidean metric in polar variables, can be supposed to describe distances on the surface of a cone. The cone has a singularity at its tip $r = 0$, except in the limiting case when the cone opens out as a plane. In this situation θ has a periodicity 2π, so one may say that the conical singularity is avoided by making θ an angular variable with this period. This is relevant for black holes because such a singularity tends to arise there. One passes to imaginary time and writes the metric for a constant θ, ϕ surface as

$$ds^2 = \left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)} = \Omega(\rho)\,(d\rho^2 + \rho^2 d\tau^2)\,, \tag{11}$$

where $\tau = \alpha t$ with the constant α so chosen as to make the conformal factor Ω finite at the horizon. For consistency, one requires

$$\rho = e^{\alpha r_*}, \tag{12}$$

where r_* is defined by

$$dr_* = \frac{dr}{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)}. \tag{13}$$

Near the horizon,

$$r_* \approx \frac{r_+^2}{r_+ - r_-}\log\left(r - r_+\right), \tag{14}$$

so that

$$\rho \approx (r - r_+)^{\alpha r_+^2/(r_+ - r_-)}, \tag{15}$$

which implies that ρ vanishes at the horizon, and

$$\Omega = \frac{\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)}{\alpha^2\rho^2} \tag{16}$$

can be made finite at the horizon by making ρ^2 vanish linearly as $r \to r_+$, *i.e.*, by choosing α to satisfy

$$\alpha r_+^2 \quad \frac{1}{2}, \tag{17}$$

i.e.,

$$\alpha = \kappa \ . \tag{18}$$

Now for the conical singularity to be avoided, one must have a periodicity of 2π for τ, *i.e.*, a periodicity for t given by $\frac{2\pi}{\alpha}$ or $\frac{2\pi}{\kappa}$. This periodicity corresponds to a *temperature*

$$T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar\,(r_+ - r_-)}{4\pi r_+^2} \ . \tag{19}$$

The periodicity can also be seen by considering an analogue of Kruskal coordinates for these black holes : the new coordinates have to be such that the metric components are nonsingular at the horizon. One first writes the metric of a surface with constant θ, ϕ in the double null form

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)(dt^2 - dr_*^2) \\ &= -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)dv\,d\omega \end{aligned} \tag{20}$$

with

$$v = t - r_* \ , \qquad \omega = t + r_* \ . \tag{21}$$

Here r is understood to be implicitly defined by using $\omega - v = 2r_*$ and the relation between r_* and r. One passes to new null coordinates $\bar{v}$ and $\bar{\omega}$ defined by

$$v = f(\bar{v}) \ , \quad \omega = g(\bar{\omega}) \ , \tag{22}$$

with appropriate functions f, g. The metric becomes

$$ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)\frac{df}{d\bar{v}}\frac{dg}{d\bar{\omega}}\,d\bar{v}d\bar{\omega} \tag{23}$$

and r is understood to be determined implicitly by

$$g(\bar{\omega}) - f(\bar{v}) = 2\,r_* . \tag{24}$$

The functions f, g are to be chosen in such a way that the coefficient of $d\bar{v}d\bar{\omega}$ in the right hand side of (23) is regular at the horizon.

The horizon corresponds to $r_* \to -\infty$ and therefore either v or $-\omega$ has to be infinite. The new coordinates are defined by [4]

$$v = f(\bar{v}) = -\frac{1}{\alpha}\log\bar{v} \ , \quad \omega = g(\bar{\omega}) = \frac{1}{\alpha}\log\bar{\omega} \ , \tag{25}$$

with the constant α to be determined. This definition has the result that one of the new coordinates has to vanish at the horizon. If we consider a point where $\bar{\omega}$ vanishes, we see that the factor $\frac{dg}{d\bar{\omega}} \propto \frac{1}{\bar{\omega}}$ in (23) becomes infinite and may make the product with $(1 - \frac{r_+}{r})$ finite. For this to happen, $\bar{\omega}$ must vanish linearly as $r \to r_+$. Now (14) indicates that

$$\frac{1}{\alpha}\log\bar{\omega} = \frac{2r_+^2}{r_+ - r_-}\log(r - r_+)\,, \tag{26}$$

so that the condition for linearity of $\bar{\omega}$ is the same as (17), *i.e.*, $\alpha = \kappa$ once again. This fixation of α completes the definition of the new coordinates. We have arranged the regularity of the metric components at vanishing $\bar{\omega}$, but with this choice one can also check that there is no problem in the region of vanishing $\bar{v}$.

Now we have to find the temperature. A time coordinate can be defined in terms of the new null variables and a vacuum can be defined in terms of this time. Physical quantities involve the new coordinates $\bar{v}, \bar{\omega}$ (and θ, ϕ). As $\bar{v} = e^{-\alpha v}$, $\bar{\omega} = e^{\alpha\omega}$ involve $e^{\alpha t}$, it is clear that the *imaginary* time has a period $\frac{2\pi}{\alpha}$, which corresponds to the same temperature as found above by the conical singularity argument.

It is to be noted that the temperature involves Planck's constant as a factor. Quantum theory causes dramatic changes in the behaviour of matter in black hole spacetimes. A scalar field theory in the background of a black hole indicates the occurrence of radiation of particles [5] at a temperature

$$T = \frac{\hbar\kappa}{2\pi}\,. \tag{27}$$

This reconfirms the connection of the surface gravity with temperature, and indeed, of the laws of black hole mechanics with thermodynamics. The scale factor $\frac{h}{2\pi}$ involves Planck's constant and is a quantum effect. Note that the first law of black hole mechanics fixes the scale of the entropy to be

$$S = \frac{A}{4\hbar}\,. \tag{28}$$

2.3 Action :

In an alternative approach to the same area formula, the grand partition function is used. For a charged black hole [6] it can be related to the classical action by

$$Z_{\text{grand}} = e^{-\frac{M - TS - \phi Q}{T}} \approx e^{-I/\hbar}\,, \tag{29}$$

where the functional integral over all configurations with the proper boundary condition is semiclassically approximated by the weight factor with the classical action I.

To evaluate the action, let us consider a euclidean Reissner – Nordström black hole in a manifold $\mathcal{M}$ with a boundary which is subsequently taken to infinity. The action has the expression

$$I = -\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{g}R + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\,\sqrt{\gamma}\,(K - K_0) +$$

$$\frac{1}{16\pi}\int_{\mathcal{M}} d^4x\,\sqrt{g}F_{\mu\nu}F^{\mu\nu}\,. \tag{30}$$

Here γ is the induced metric on the boundary $\partial \mathcal{M}$ and K the extrinsic curvature, from which a subtraction has to be made to make the action finite. This form of the action leads to the Einstein – Maxwell equations of motion if variations are carried out with $g_{\mu\nu}$, A_μ fixed on the boundary. This boundary condition implies that the temperature and the electrostatic potential are fixed on the boundary and is thus appropriate for the grand canonical ensemble.

The first term of the action vanishes because Einstein's equations lead to $R = 0$.

To evaluate the second term, we take the boundary of the manifold at $r = r_B \to \infty$. Then

$$\begin{aligned} K &= -\frac{1}{\sqrt{g_{tt}}\,r^2}\frac{1}{\sqrt{g_{rr}}}\frac{d}{dr}(\sqrt{g_{tt}}\,r^2) \\ &= -\frac{1}{r^2}\frac{d}{dr}[(1-\frac{M}{r}+\ldots..)r^2] \\ &= -\frac{1}{r^2}\frac{d}{dr}(r^2 - Mr) \end{aligned} \tag{31}$$

and

$$\int d^3x\,\sqrt{\gamma} = \int dt\,(1-\frac{M}{r}+\ldots.)\,4\pi r^2. \tag{32}$$

We see that $\int d^3x\,\sqrt{\gamma}K$ diverges as $r \to \infty$, but this can be cured by subtracting from K the flat space contribution $K_0 = -\frac{1}{r^2}\frac{d}{dr}r^2$. The second piece of the action becomes

$$\begin{aligned} &-\frac{1}{8\pi}\int dt\left(1-\frac{M}{r}+\ldots\right)4\pi r^2\,\frac{1}{r^2}\frac{d}{dr}(-Mr)\,|_{r=r_B\to\infty} \\ &= -\frac{1}{2}\int dt(-M) = \frac{1}{2}\beta M. \end{aligned} \tag{33}$$

Finally, the third term of the action becomes

$$\begin{aligned} &-\frac{1}{16\pi}\int dt\cdot 4\pi\int dr^2.2.\frac{Q^2}{r^4} \\ &= -\frac{1}{2}\int dt\frac{Q^2}{r_+} \\ &= -\frac{1}{2}\beta Q\phi, \end{aligned} \tag{34}$$

where ϕ is the electrostatic potential at the horizon. The sign is negative here because in the euclidean solution the electric field is purely imaginary. Putting all pieces of the action together, we find the numerical value of the action to be

$$I = \frac{1}{2}\beta\,(M - Q\phi) = \frac{A}{4} \tag{35}$$

Consequently,

$$M = T\,(S + \frac{A}{4\hbar}) + \phi Q\,. \tag{36}$$

Now, there is a standard formula named after Smarr [7],

$$M = \frac{\kappa A}{4\pi} + \phi Q\,, \tag{37}$$

which can be rewritten as

$$M = T\,\frac{A}{2\hbar} + \phi Q\,. \tag{38}$$

Comparison with (36) indicates once again the relation (28). Although the result is the same, it should be noted that there is a new input : the functional integral. This opens the possibility that corrections to the above formulas may be obtained by improving the approximation used in the calculation of the functional integral.

2.4 The classical extremum of the action :

It is interesting to study the extremum of the action corresponding to a classical black hole. The action for the euclidean version of a Reissner – Nordström black hole on a four dimensional manifold $\mathcal{M}$ with a boundary has been given in (30). Let us restrict ourselves to spherically symmetric metrics [8] on $\mathcal{M}$:

$$ds^2 = b^2 d\tau^2 + \alpha^2 dr^2 + r^2 d\Omega^2\,, \tag{39}$$

with the variable r ranging between r_+ (the horizon) and r_B (the boundary), and b, α functions of r only. There are boundary conditions on these :

$$b(r_+) = 0,\ \ 2\pi b\,(r_B) = \beta\,. \tag{40}$$

Here β is the inverse temperature and r_B the radius of the boundary which will be taken to infinity. There is another boundary condition involving $b'(r_+)$: It reflects the non-extremal nature of the black hole :

$$\frac{b'(r_+)}{\alpha(r_+)} = 1\,. \tag{41}$$

The vector potential is taken to be zero and the scalar potential satisfies the boundary conditions

$$A_\tau(r_+) = 0\,,\quad A_\tau(r_B) = \frac{\beta\Phi}{2\pi i}\,. \tag{42}$$

The action (30) with this form of the metric depends on the functions $b(r)$, $\alpha(r)$ and $A_\tau(r)$: this may be regarded as a reduced action. Variation of these functions with proper boundary conditions leads to reduced versions of the Einstein – Maxwell equations. The solution of a subset of these equations, namely the Gauss law and the Hamiltonian constraint, is given by [8]

$$\frac{1}{\alpha} = \left[1 - \frac{2m}{r} + \frac{q^2}{r^2}\right]^{1/2},\quad A'_\tau = -\frac{iqd\alpha}{r^2}\,, \tag{43}$$

with the mass parameter m and the charge q arbitrary. The reason why these parameters have not been written here as functions of β, Φ is that some of the equations of motion and the corresponding boundary conditions have not yet been imposed on the solution. Instead of that, the action may be expressed in terms of m, q and then extremized with respect to m, q [8] to yield the classical black hole configuration.

The value of the action for a given set of m, q with $r_B \to \infty$ is [8]

$$I = \beta(m - q\Phi) - \pi(m + \sqrt{m^2 - q^2})^2 . \tag{44}$$

Thus the partition function is of the form

$$\int d\mu\,(m) \int d\mu\,(q)\, e^{-I(q,m)}, \tag{45}$$

with I given by (44). The semiclassical approximation involves replacing the double integral by the maximum value of the integrand, *i.e.*, by the exponential of the negative of the minimum I. We consider the variation of I as q, m vary. It is clear from (44) that if the action is extremized with respect to q, m, the result, a function of β, Φ, is

$$I = \frac{\beta^2(1-\Phi^2)^2}{16\pi}. \tag{46}$$

The corresponding values of q, m, which can be seen to agree with the classical values, are given by

$$q = \frac{\beta\Phi(1-\Phi^2)}{4\pi}, \tag{47}$$

$$r_+ = m + \sqrt{m^2 - q^2} = \frac{\beta(1-\Phi^2)}{4\pi}. \tag{48}$$

This leads to an entropy equal to

$$\begin{aligned} S &= \beta^2 \frac{\partial(\beta^{-1} I)}{\partial\beta} \\ &= \frac{\beta^2(1-\Phi^2)^2}{16\pi} \\ &= \pi r_+^2 , \end{aligned} \tag{49}$$

i.e., a quarter of the area. The averages Q, M, as opposed to the parameters q, m, are supposed to be calculated from β, Φ, but in the semiclassical approximation where the functional integral is saturated by just the classical configuration, agree with the values of q, m at the extremum.

It is noteworthy that in the above extremization, the values of q, m obtained above correspond to a saddle point but not a true minimum. This suggests instability. However, stability of the extremum configurations can be achieved [8] with finite r_B.

2.5 *Matter outside black hole :*

The investigation of field theory in the background of a black hole [5] had given a physical meaning to the temperature of a black hole but the entropy remained mysterious. An attempt was then made [9] to study the entropy of matter in such a background. He introduced the brick-wall model, where the wave function is cut off just outside the horizon. Mathematically,

$$\varphi(x) = 0 \quad \text{at } r = r_h + \varepsilon \tag{50}$$

where ε is a small, positive, quantity and signifies an ultraviolet cut-off. There is also an infrared cut-off

$$\varphi(x) = 0 \quad \text{at } r = L, \tag{51}$$

with $L >> r_h$.

We consider a static, spherically symmetric black hole spacetime with the metric

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)\, d\Omega^2 , \tag{52}$$

and study spinless particles bounded by the brick wall at $r = r_h + \varepsilon$ and the long distance cut-off at $r = L$. An r- dependent radial wave number can be introduced for particles with mass m, energy E and orbital angular momentum l by

$$k_r^2(r, l, E) = g_{rr}\,[-g^{tt}E^2 - l(l+1)\, g^{\theta\theta} - m^2]. \tag{53}$$

Only such values of E are to be considered here that the above expression is non-negative. The values are further restricted by the semiclassical quantization condition

$$n_r\pi = \int_{r_h+\varepsilon}^{L} dr\, \kappa_r\,(r, l, E), \tag{54}$$

where the radial quantum number n_r has to be a non-negative integer.

The free energy F at inverse temperature β is given by a sum over single particle states

$$\begin{aligned}
\beta F &= \sum_{n_r, l, m_l} \log(1 - e^{-\beta E}) \\
&\approx \int dl\,(2l+1) \int dn_r\, \log(1 - e^{-\beta E}) \\
&= -\int dl\,(2l+1) \int d(\beta E)\,(e^{\beta E} - 1)^{-1} n_r \\
&= -\frac{\beta}{\pi} \int dl\,(2l+1) \int dE\,(e^{\beta E} - 1)^{-1} \int_{r_h+\varepsilon}^{L} dr\, g_{rr}^{1/2} \\
&\qquad \sqrt{-g^{tt}\, E^2 - l(l+1)g^{\theta\theta} - m^2} \\
&= -\frac{2\beta}{3\pi} \int_{r_h+\varepsilon}^{L} dr\, g_{rr}^{1/2} g_{\theta\theta}\, (-g_{tt})^{-3/2} \\
&\qquad \int dE\,(e^{\beta E} - 1)^{-1}\,[E^2 + g_{tt}m^2]^{3/2}.
\end{aligned} \tag{55}$$

Here the limits of integration for l, E are such that the arguments of the square roots are nonnegative. The l integration is straightforward and has been explicitly carried out. The E integral can be evaluated only approximately.

Because of the asymptotically flat nature of the metric of the spacetime containing the black hole, the contribution to the r integral from large values of r corresponds to the expression for the free energy valid in flat spacetime :

$$F_0 = -\frac{2}{9\pi} L^3 \int_m^\infty dE \frac{(E^2 - m^2)^{3/2}}{e^{\beta E} - 1}. \tag{56}$$

This piece is not relevant for us. The contribution of the black hole is singular in the limit $\varepsilon \to 0$. The leading singularity is obtained by taking the metric coefficients, multiplied by appropriate powers of $(r - r_h)$ to make them finite at the horizon, outside the integral. For an ordinary *i.e.*, *non-extremal* black hole, g_{rr} has a linear singularity and g_{tt} a linear zero at $r = r_h$, while $g_{\theta\theta}$ is regular there. Thus,

$$F_{sing} \approx -\frac{2\pi^3}{45\varepsilon\beta^4} [(r - r_h) g_{rr}]^{1/2} \left(-\frac{g_{tt}}{r - r_h}\right)^{-3/2} g_{\theta\theta}|_{r=r_h}, \tag{57}$$

where the lower limit of the E integral has been approximately set equal to zero. If the proper value is taken, there are corrections involving $m^2\beta^2$ which will be ignored here.

The entropy due to the black hole can be obtained from the formula

$$S = \beta^2 \frac{\partial F}{\partial \beta}. \tag{58}$$

This gives

$$S_{sing} = \frac{8\pi^3}{45\beta^3\varepsilon} [(r - r_h) g_{rr}]^{1/2} \left(-\frac{g_{tt}}{r - r_h}\right)^{-3/2} g_{\theta\theta}|_{r=r_h}. \tag{59}$$

To put this expression in a more familiar form, we write the temperature in terms of the surface gravity as

$$\begin{aligned} \frac{1}{\beta} &= \frac{1}{2\pi} (g_{rr})^{-1/2} \frac{\partial}{\partial r} (-g_{tt})^{1/2} |_{r=r_h} \\ &= \frac{1}{4\pi} (g_{rr})^{-1/2} (-g_{tt})^{1/2} \frac{\partial}{\partial r} (-g_{tt}) |_{r=r_h} \\ &= \frac{1}{4\pi} [(r - r_h) g_{rr}]^{-1/2} \left(-\frac{g_{tt}}{r - r_h}\right)^{1/2} |_{r=r_h}. \end{aligned} \tag{60}$$

It is also necessary to measure the width of the brick wall in terms of the proper radial variable $\tilde{r}$ defined by $d\tilde{r}^2 = g_{rr} dr^2$:

$$\tilde{\varepsilon} = \tilde{r}(r_h + \varepsilon) - \tilde{r}(r_h) \approx 2\varepsilon^{1/2} [(r - r_h) g_{rr}]^{1/2} |_{r=r_h}. \tag{61}$$

On making these substitutions, we find

$$S_{sing} = \frac{1}{90\tilde{\varepsilon}^2} g_{\theta\theta}|_{r=r_h} = \frac{1}{360\pi\tilde{\varepsilon}^2} \text{Area}, \tag{62}$$

in agreement with known special cases [9].

It should be emphasized that the above derivation crucially depends on the behaviour of the metric coefficients near the horizon.

The appearance of the area in this and similar calculations led to a lot of interest. The divergence in the limit of vanishing cut off ε is clearly due to the concentration of the matter states near the horizon. The question then was how the finite result $A/(4[G]\hbar)$ is to be obtained from this expression. It was suggested that different species of matter, which have to be summed over, might renormalize the Newton constant in the denominator of the entropy and might even produce all of it ! However, an alternative point of view is that the calculation indicated above refers to the entropy of the matter rather than that of the black hole. Whereas the calculation of the temperature of matter can tell us about the temperature of the black hole, the entropy of one need not have any connection with that of the other. The two simply have to be added up in this view.

3. Extremal case

In the recent past, there has been special interest in *extremal* black holes. There are so many special properties that have been observed for these black holes that it is often suspected that the topological differences between extremal and non-extremal black holes (seen in their euclidean versions) may lead to a discontinuity in the extremal limit of non-extremal black holes. We shall see in the following whether or in what sense there is a discontinuity.

3.1 Temperature :

We have considered above the occurrence of a conical singularity in the non-extremal case. In the extremal case, things are very different. Near the horizon, now, instead of (14), one finds

$$r_* \approx -\frac{M^2}{r-M}, \tag{63}$$

so that $\rho = e^{\alpha r_*}$ has an essential singularity as $r \to r_+$. Then there is no value of the constant α which can make the conformal factor $\Omega = \frac{(1-\frac{r_+}{r})(1-\frac{r_-}{r})}{\alpha^2\rho^2}$ in (11) regular at the horizon. This means that the extremal metric is not of the form (10) and there is no *conical* singularity. Consequently, the derivation of a *conical* temperature fails. However, it should be noted that there is definitely a singularity of some other kind.

We consider again the question of an analogue of Kruskal coordinates. As in the conical approach, the case of the extremal black hole is very different from the non-extremal cases. The behaviour (63) of r_* near the horizon is linearly divergent and $\overline{\omega}$ has an essential singularity. So the logarithmic transformation (25) cannot, for any choice of α, lead to metric components regular at the horizon. A transformation that does has been known for a long time [10]:

$$\upsilon = f(\overline{\upsilon}) = M \tan \overline{\upsilon}, \;\; \omega = g(\overline{\omega}) = M \cot \overline{\omega}. \tag{64}$$

It is not difficult to check the regularity of the coefficient in (23). But the new coordinates do not exhibit any periodicity in imaginary time, so that the vacuum corresponding to the time defined by the above coordinates is *not thermal*.

3.2 Vanishing action

The classical action can be seen to *vanish* in the extremal case, and is thus discontinuous from the non-extremal one, which is equal to the area. Let us once again consider the Euclidean Reissner – Nordström black hole in the manifold with boundary.

The first term of the action (30) vanishes again because Einstein's equations lead to $R = 0$. As before, the Second piece of the action is $\frac{1}{2}\beta M$. The third term of the action is

$$\begin{aligned} & -\frac{1}{16\pi}\int dt.4\pi\int dr^2.2.\frac{Q^2}{r^4} \\ = & -\frac{1}{2}\int dt\frac{Q^2}{r_+} \\ = & -\frac{1}{2}\frac{\beta M^2}{r_+} \\ = & -\frac{1}{2}\beta M \end{aligned} \tag{65}$$

as $Q = M = r_+$ here.

Putting all pieces of the action together, we find the value of the action to be

$$I = \frac{1}{2}\beta(M - M) = 0 . \tag{66}$$

In doing this calculation, β has been assumed finite. If the extremal *limit* of a non-extremal black hole is taken, this quantity actually goes to infinity. However, as we have seen, there is no conical singularity in the extremal case, so that there is no reason to fix the temperature in this case, and the temperature should be regarded as arbitrary [11].

3.3 Entropy :

We have seen above how non-extremal black holes may be said to possess an entropy proportional to the area of the horizon. If one is interested in an extremal black hole, one may be tempted to regard it as a special limiting case of a sequence of non-extremal black holes and thus infer that the same formula should hold for the entropy. But it was pointed out in the context of Reissner – Nordström black holes [11] that the extremal and non-extremal cases of the Euclidean version are topologically different, so that continuity need not hold. Indeed, it was shown in [12] that the derivation of an expression for the thermodynamic entropy of an extremal black hole following [6] yields not the area but essentially zero – as one might have expected from the value of the action – together with an extra term proportional to the mass of the black hole.

For a charged black hole, the first law of thermodynamics

$$TdS = dM - \Phi dQ , \tag{67}$$

involves two "intensive" variables, *viz.* T, the temperature, conjugate to M, and Φ the chemical potential, conjugate to Q. We are interested in the extremal case $Q = M$. Then there is only one independent thermodynamical variable, Q or M, so the first law should involve only one conjugate variable and can be written as

$$dS = \gamma dM . \tag{68}$$

If this equation is sought to be understood in terms of the previous one, γ must be interpreted as $\frac{1-\Phi}{T}$ as (see below).

To understand the meaning of γ, one has to imagine a thermodynamic system of mass M and charge Q in contact with a reservoir of energy and charge such that exchanges of energy and charge with the system are always constrained to be equal. In this situation, the total change of entropy of the system and the reservoir is given by

$$dS_{tot} = \gamma \, dM - \frac{dM}{T_{reservoir}} + \frac{\Phi_{reservoir} \, dM}{T_{reservoir}} . \tag{69}$$

The condition for equilibrium is then

$$\gamma = \frac{1-\Phi_{reservoir}}{T_{reservoir}} . \tag{70}$$

Thus, instead of the usual equality of temperatures and chemical potentials, there is only *one* condition, with γ equalling a certain combination of the temperature and the chemical potential of the reservoir. In other words, one cannot even talk separately of a temperature and a chemical potential for the system : there is only the combination γ. Correspondingly, the ensemble is not quite grand canonical, but a *reduced* grand canonical one.

Indeed, the partition function also has to be written as

$$Z = e^{S-\gamma M} = e^{-I} . \tag{71}$$

Here, I is the effective action. If it is set equal to the classical on-shell action in the lowest approximation, it vanishes in the extremal case, as seen above. This implies

$$S = \gamma M. \tag{72}$$

Comparison with the first law (68) then shows that

$$d\gamma = 0, \tag{73}$$

so that γ is a constant, hence the entropy is a constant times the mass. This constant may of course vanish, but that is a special case. Thus there is no area term, which is natural considering that the action is *not* given by the area but vanishes. But the entropy may still be nonzero and is allowed to be proportional to the mass.

All this assumes that as in the non-extremal case here too the effective action may be approximated by the classical action. However, it should be pointed out that the zero value cannot be an isolated minimum of the action : the value being independent of the mass of the black hole, there may be uncontrolled fluctuations. So it is necessary to be cautious about the use of the formula $S = \gamma M$. In fact, in the case of black holes which have nonzero area in the extremal situation, this contribution may be swamped by an area contribution from non-extremal configurations which have to be taken into account in the sum over topologies – extremal and non-extremal – in the functional integral [13]. The formula $S = \gamma M$ may be relevant in the case of extremal black holes with zero area [14].

3.4 The area law again :

Usually, when one quantizes a classical theory, one tries to preserve the classical topology. In this spirit, one usually seeks to have a quantum theory of extremal black holes based exclusively on extremal topologies. As an alternative, one can consider a quantization where a sum over topologies is carried out. Thus, in our consideration of the functional integral, classical configurations corresponding to both topologies will be included. The extremality condition will then be imposed on the averages that result from the functional integration. We shall, following [6] and [8], use a grand canonical ensemble. Here, the temperature and the chemical potential are supposed to be specified as inputs, and the average mass M and charge Q of the black hole are outputs. So the actual definition of extremality that we have in mind for a Reissner- Nordström black hole is $Q = M$. This may be described as *extremalization after quantization*, as opposed to the usual approach of *quantization after extremalization* [13].

The action for the euclidean version of a Reissner – Nordström black hole on a four dimensional manifold $\mathcal{M}$ with a boundary has been given in (30). The class (39) of spherically symmetric metrics [8] is considered on $\mathcal{M}$ with the variable r ranging between r_+ (the horizon) and r_B (the boundary), and b, α functions of r only. There are the boundary conditions (40) as usual. β is the inverse temperature and r_B the radius of the boundary which will be taken to infinity. There is another boundary condition involving $b'(r_+)$. It reflects the extremal / non-extremal nature of the black hole and is therefore different for the two conditions : In the non-extremal case, it is given by (41), and in the extremal case,

$$\frac{b'(r_+)}{\alpha(r_+)} = 0 \text{ in extremal case.} \tag{74}$$

The vector potential is taken to be zero and the scalar potential satisfies the boundary conditions (42).

The action (30) with the form (39) of the metric depends on the functions $b(r)$, $\alpha(r)$ and $A_\tau(r)$. Variation of these functions with proper boundary conditions leads to reduced versions of the Einstein – Maxwell equations. The solution of a subset of these equations, namely the Gauss law and the Hamiltonian constraint, is given by (43) with the mass parameter m and the charge q arbitrary. The reason why these parameters are not expressed as functions of β, Φ is that some of the equations of motion and the corresponding boundary conditions have not been imposed on the solution. Instead of that, the action is to be expressed in terms of m, q and then extremized with respect to m, q.

The value of the action corresponding to the solution depends on the boundary condition :

$$\begin{aligned} I &= \beta(m - q\Phi) - \pi(m + \sqrt{m^2 - q^2}\,)^2 \text{ for non-extremal bc,} \\ I &= \beta(m - q\Phi) \text{ for extremal } bc. \end{aligned} \tag{75}$$

The first line is as in (44), where the non-extremal boundary condition was used, while the second line corresponds to the extremal boundary condition used in connection with a semiclassically quantized extremal black hole [15]. As the euclidean topologies of non-extremal and extremal black holes are different, quantization was done separately for the two cases in [8, 15]. The topology was selected before quantization.

As indicated above, a different approach is to be used here. The two topologies are to be summed over in the functional integral [13] and the extremality condition imposed afterwards. Thus the partition function is of the form

$$\sum_{\text{topologies}} \int d\mu(m) \int d\mu(q) e^{-I(q,m)}, \tag{76}$$

with I given by (75) as appropriate for non-extremal/extremal q. The semi-classical approximation involves replacing the double integral by the maximum value of the integrand, *i.e.*, by the exponential of the negative of the minimum I. We consider the variation of I as q, m vary in both topologies. It is clear from (75) that the non-extremal action is lower than the extremal one for neighbouring values of q, m. Consequently, the partition function is to be approximated by $e^{-I_{min}}$, where I_{min} is the classical action for the *non-extremal* case, *minimized* with respect to q, m. The result, which should be a function of β, Φ, can be read off [8]. It leads to an entropy equal to a quarter of the area for all values of β, Φ. The averages Q, M, as opposed to the parameters q, m, are calculated from β, Φ. We are interested in $|Q| = M$, *i.e.*, the extremal black hole. This is obtained for limiting values

$$\beta \to \infty, \ |\Phi| \to 1, \ \text{with } \beta(1-|\Phi|) \text{ finite} \tag{77}$$

of the ensemble parameters and is described by the effective action

$$I = \pi M^2 = \frac{(\beta(1-|\Phi|))^2}{4\pi} \tag{78}$$

It is worth repeating that for extremal black holes, the parameters β, Φ necessarily enter in the combination $\gamma \equiv \beta(1-|\Phi|)$. This combination does occur here as it also does in the case with purely extremal topology [15].

Thus in the limit, the partition function takes the form

$$Z = e^{-\frac{\gamma^2}{4\pi}} = e^{-\pi M^2} = e^{\frac{A}{4}} \tag{79}$$

This continues to correspond to an entropy of a quarter of the area of the horizon. Hence, in this approach, extremal black holes too satisfy the area law.

3.5 *Entropy of matter :*

Finally, we comment on the entropy of matter in the background of an extremal black hole. For the extremal dilatonic black hole, described by the metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r(r-a)d\Omega \tag{80}$$

with $a = 2M$, F has a logarithmic singularity. This is present in the non-extremal case as well, but it is in general ignored because of the presence of a linearly divergent term. However, the linear term vanishes when $a = 2M$, *i.e.*, when the black hole becomes extremal. In this case, the logarithmic term is the dominant one. It arises because $g_{\theta\theta}$ vanishes linearly at the horizon and has to be kept inside the r integral in the last line of (55). One obtains

$$F_{dil}^{ex} \approx -\frac{\pi^3}{45M} \log\left(\frac{2M}{\varepsilon}\right)\left(\frac{2M}{\beta}\right)^4 \tag{81}$$

in the same approximation as above. Correspondingly,

$$S_{dil}^{ex} = \frac{1}{360} 2 \log \frac{(4M)^2}{\varepsilon^2} . \tag{82}$$

As the area of the horizon vanishes, one might have expected the entropy to vanish altogether. What does happen is that the linear divergence vanishes, but the logarithmic divergence, which is of course weaker, stays on [16].

For an extremal Reissner – Nordström black hole, the singularity of the r integral in the last line of (55) becomes stronger because g_{rr} has now a quadratic singularity and g_{tt} a quadratic zero at the horizon. One obtains

$$F_{RN}^{ex} \approx -\frac{2\pi^3 r^2}{135\varepsilon^3}\left(\frac{r_+}{\beta}\right)^4 . \tag{83}$$

The contribution to the entropy due to the presence of the black hole is

$$S_{RN}^{ex} = \frac{8\pi^3}{135}\left(\frac{r_+}{\beta}\right)^3\left(\frac{r_+}{\varepsilon}\right)^3 . \tag{84}$$

The formula for the Hawking temperature of the Reissner–Nordström black hole vanishes in the extremal case. If the vanishing temperature is inserted, the expression (84) for the entropy also vanishes and this was the understanding about this entropy until recently. However, the temperature may be non-vanishing, [11], as mentioned above. For a general β, (84) is nonzero and nominally cubically divergent, whereas non-extremal black holes have only a linear divergence in terms of the cutoff ε. In this extremal case, the proper radial variable $\tilde{r}$ defined by $d\tilde{r}^2 = g_{rr}\, dr^2$ goes like

$$\tilde{r} = r + r_+ \log\frac{r - r_+}{r_+} . \tag{85}$$

The horizon corresponds to $\tilde{r} = -\infty$, so that the cutoff is at a *large* negative distance $\tilde{r} = -\Lambda$, with

$$-\Lambda = r_+ + \varepsilon + r_+ \log\frac{\varepsilon}{r_+} . \tag{86}$$

This means that (84) goes like $\exp\frac{3\Lambda}{r_+}$, so that the true divergence is exponential, *i.e.*, much stronger than in the non-extremal case [12].

These results indicate that contrary to the case of non-extremal black holes, here the entropy of matter does not behave at all like the entropy of the gravitational field. This would seem to suggest that the entropy of matter in the background of a black hole has little to do with the entropy of the black hole itself, contrary to earlier expectations based on studies of non-extremal black holes.

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